

# BERNSTEIN-WALSH INEQUALITIES IN HIGHER DIMENSIONS OVER EXPONENTIAL CURVES

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ABSTRACT. Let  $\mathbf{x} = (x_1, \dots, x_d) \in [-1, 1]^d$  be linearly independent over  $\mathbb{Z}$ , set  $K = \{(e^z, e^{x_1 z}, e^{x_2 z} \dots, e^{x_d z}) : |z| \leq 1\}$ . We prove sharp estimates for the growth of a polynomial of degree  $n$ , in terms of

$$E_n(\mathbf{x}) := \sup\{\|P\|_{\Delta^{d+1}} : P \in \mathcal{P}_n(d+1), \|P\|_K \leq 1\},$$

where  $\Delta^{d+1}$  is the unit polydisk. For all  $\mathbf{x} \in [-1, 1]^d$  with linearly independent entries, we have the lower estimate

$$\log E_n(\mathbf{x}) \geq \frac{n^{d+1}}{(d-1)!(d+1)} \log n - O(n^{d+1});$$

for Diophantine  $\mathbf{x}$ , we have

$$\log E_n(\mathbf{x}) \leq \frac{n^{d+1}}{(d-1)!(d+1)} \log n + O(n^{d+1}).$$

In particular, this estimate holds for almost all  $\mathbf{x}$  with respect to Lebesgue measure. The results here generalize those of [CP10] for  $d = 1$ , without relying on estimates for best approximants of rational numbers which do not hold in the vector-valued setting.

## 1. INTRODUCTION

For any  $\ell \in \mathbb{N}$  we let  $\Delta^\ell$  denote the unit polydisk

$$\{\mathbf{z} = (z_1, z_2, \dots, z_\ell) \in \mathbb{C}^\ell : |z_i| \leq 1, \forall i = 1, 2, \dots, \ell\}.$$

For a given  $d \in \mathbb{N}$  we consider a vector  $\mathbf{x} = (x_1, \dots, x_d) \in [-1, 1]^d$  and a compact set

$$K = K(\mathbf{x}) = \{(e^z, e^{x_1 z}, e^{x_2 z} \dots, e^{x_d z}) : |z| \leq 1\}.$$

For any  $n, \ell \in \mathbb{N}$  we let  $\mathcal{P}_n(\ell)$  denote the subspace of polynomials  $P \in \mathbb{C}[z_1, \dots, z_\ell]$  of degree  $n$ . For any subset  $D \subset \mathbb{C}^\ell$  and polynomial  $P$  we define  $\|P\|_D = \{|P(\mathbf{z})| : \mathbf{z} \in D\}$ . We note that  $\|\cdot\|_K$  defines a norm only if  $\{1, x_1, x_2, \dots, x_d\}$  are linearly independent over  $\mathbb{Z}$  which is what we will assume throughout the paper. For any  $n \in \mathbb{N}$  we let

$$E_n(\mathbf{x}) := \sup\{\|P\|_{\Delta^{d+1}} : P \in \mathcal{P}_n(d+1), \|P\|_K \leq 1\}.$$

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From the equivalence of the norms  $\|\cdot\|_{\Delta^{d+1}}$  and  $\|\cdot\|_K$  we see (c.f. [CP03]) for any  $\mathbf{z} = (z_0, z_1, \dots, z_d) \in \mathbb{C}^{d+1}$  that

$$(1) \quad |P(\mathbf{z})| \leq \|P\|_K E_n(\mathbf{x}) \exp(n \log^+ \max\{|z_0|, \dots, |z_d|\}).$$

Let  $e_n(\mathbf{x}) = \log E_n(\mathbf{x})$ . On  $\mathbb{R}^d$ , we fix the maximum norm  $\|\cdot\|$  given by  $\|\mathbf{x}\| = \max_{1 \leq \ell \leq d} |x_\ell|$ . For any  $x \in \mathbb{R}$  we let  $\langle x \rangle$  denote the distance from  $x$  to the nearest integer, that is,  $\langle x \rangle = \min\{|x - k| : k \in \mathbb{Z}\}$ . We say that a vector  $\mathbf{x} \in \mathbb{R}^d$  is *Diophantine* if there exist  $\mu \geq d$  and  $\epsilon > 0$  such that for any  $\mathbf{q} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$  we have  $\langle \mathbf{q} \cdot \mathbf{x} \rangle > \epsilon \|\mathbf{q}\|^{-\mu}$ . From Dirichlet's approximation theorem (see e.g. [Sc80]) we know that there are no Diophantine vectors with  $\mu < d$ . When  $d = 1$ , it was shown in [CP10] that if  $x \in \mathbb{R}$  is Diophantine then the exponent  $e_n(x)$  grows like  $\frac{1}{2}n^2 \log n$ . Our goal in this paper is to generalise this result for any  $d \in \mathbb{N}$ . Using the existence of exponential polynomials in  $\mathcal{P}_n(d+1)$  with a zero of order at least  $\deg \mathcal{P}_n - 1$  we get the following.

**Theorem 1.1.** *For any  $\mathbf{x} \in \mathbb{R}^d$  with  $\{1, x_1, \dots, x_d\}$  linearly independent over  $\mathbb{Z}$  we have*

$$e_n(\mathbf{x}) \geq \frac{n^{d+1}}{(d-1)!(d+1)} \log n - O(n^{d+1}),$$

where the implied constant depends on  $\mathbf{x}$  and  $d$  only.

As for the upper estimate we get

**Theorem 1.2.** *If  $\mathbf{x} \in [-1, 1]^d$  is Diophantine then for any  $n \in \mathbb{N}$  we have*

$$(2) \quad e_n(\mathbf{x}) \leq \frac{n^{d+1}}{(d-1)!(d+1)} \log n + O(n^{d+1}),$$

where the implied constant depends on  $\mathbf{x}$  and  $d$  only. In particular, (2) holds for a.e.  $\mathbf{x} \in [-1, 1]^d$ .

To prove their result Coman and Poletsky make use of the well developed theory of continued fractions in  $\mathbb{R}$ . As there is no good analogue of continued fractions theory in higher dimensions we will consider a different approach.

We say that a vector  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$  with  $\{x_1, \dots, x_d\}$  linearly independent over  $\mathbb{Q}$  is *Liouville* if it is not Diophantine, that is, for any  $n \in \mathbb{N}$  there exists  $\mathbf{q} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$  such that  $\langle \mathbf{q} \cdot \mathbf{x} \rangle < \|\mathbf{q}\|^{-n}$ . Let  $\mathcal{L}_d$  denote the set of Liouville vectors in  $\mathbb{R}^d$ . Let  $W_d(\alpha)$  denote the set of vectors  $\mathbf{x} \in \mathbb{R}^d$  such that there are infinitely many integer vectors  $\mathbf{q} \in \mathbb{Z}^d$  satisfying  $\langle \mathbf{q} \cdot \mathbf{x} \rangle < \|\mathbf{q}\|^{-\alpha}$ . It was proved in [BD86] that the Hausdorff dimension of  $W_d(\alpha)$  is  $(d-1) + \frac{d+1}{1+\alpha}$ . Since  $\mathcal{L}_d = \bigcap_{\alpha \geq d} W_d(\alpha)$ , it follows that the Hausdorff dimension of  $\mathcal{L}_d$  is at most  $d-1$ . In particular,  $\mathcal{L}_d$  has zero Lebesgue measure which justifies the last part of Theorem 1.2.

We note that for any nonzero  $\mathbf{q} \in \mathbb{Z}^d$  the set  $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{q} \cdot \mathbf{x} = 0\}$  is a hyperplane in  $\mathbb{R}^d$  and is contained in  $\mathcal{L}_d$ . Together with the above upper estimate we get that the set  $\mathcal{L}_d$  of Liouville  $d$ -vectors has Hausdorff dimension  $d-1$ .

We now turn to discuss the exceptional set of points in  $\mathbb{R}^d$  for which  $e_n(\mathbf{x})$  grows faster than  $Cn^{d+1} \log n$ . To this end, we define the set

$$W(d) = \left\{ \mathbf{x} \in [-1, 1]^d : \limsup_{\|\mathbf{q}\| \rightarrow \infty} \frac{-\log \langle \mathbf{q} \cdot \mathbf{x} \rangle}{\|\mathbf{q}\|^{d+1} \log \|\mathbf{q}\|} = \infty \right\},$$

where  $\mathbf{q} \in \mathbb{Z}_{\geq 0}^d := \{(q_1, \dots, q_d) \in \mathbb{Z}^d : q_1, \dots, q_d \geq 0\}$ .

**Theorem 1.3.** *For any  $\mathbf{x} \in W(d)$ ,  $\limsup_n \frac{e_n(\mathbf{x})}{n^{d+1} \log n} = \infty$ .*

It is easy to see (e.g. from Theorem 1.2) that  $W(d) \subset \mathcal{L}_d$  so that it has Hausdorff dimension at most  $d - 1$ . In fact, we have

**Theorem 1.4.** *Hausdorff dimension of the exceptional set  $W(d)$  is  $d - 1$ .*

It was proved in [CP10] that when  $d = 1$  the set of points  $x$  for which  $e_n(x)$  grow faster than  $\frac{1}{2}n^2 \log n$  is uncountable. For  $d > 1$ , since the Hausdorff dimension of  $W(d)$  is positive we in particular get that  $W(d)$  is uncountable. Thus, for any  $d \in \mathbb{N}$  the set of points  $\mathbf{x}$  for which  $e_n(\mathbf{x})$  grow faster than  $\frac{1}{(d-1)!(d+1)}n^{d+1} \log n$  is uncountable and has Hausdorff dimension  $d - 1$ .

In the next section we will prove Theorem 1.2 and in § 3 we obtain Theorem 1.1, Theorem 1.3, and Theorem 1.4.

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## 2. UPPER ESTIMATE

In this section our goal is to obtain Theorem 1.2. We state [CP10, Lemma 2.4]

**Lemma 2.1.** *Let  $x, y \in \mathbb{Z}$  with  $x \leq y$  be given. For any  $\alpha \in \mathbb{R}$  we have*

$$\prod_{j=x}^y |j - \alpha| \geq \langle \alpha \rangle \left( \frac{y - x}{2e} \right)^{y-x}.$$

Let  $\mathbf{x} \in \mathbb{R}^d$  and  $n \in \mathbb{N}$  be given. For any  $\ell \in \{0, 1, \dots, n\}$  and  $\mathbf{m} \in \mathbb{Z}^d$  with  $m_1, \dots, m_d \in \{0, 1, \dots, n\}$  we define

$$(3) \quad \beta(\ell, \mathbf{m}) = \prod_{j_0 + j_1 + \dots + j_d \leq n, (j_0, \mathbf{j}) \neq (\ell, \mathbf{m})} ((\ell - j_0) + (\mathbf{m} - \mathbf{j}) \cdot \mathbf{x}),$$

where each  $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d$  has nonnegative components. We will need the following estimate.

**Proposition 2.2.** *If  $\mathbf{x}$  is Diophantine, then there exists a constant  $C_{\mathbf{x}, d} > 0$  such that*

$$\log |\beta(\ell, \mathbf{m})| \geq \frac{1}{(d+1)!} n^{d+1} \log n - C_{\mathbf{x}, d} n^{d+1}.$$

To obtain the proposition we need the following lemmas. We set  $|\mathbf{j}| = j_1 + \dots + j_d$ . Arguing inductively on  $d$  it is easy to see that

**Lemma 2.3.** *For any  $m \in \mathbb{N}$ , the set  $\{\mathbf{j} \in \mathbb{Z}^d : |\mathbf{j}| = m, j_1, \dots, j_d \geq 0\}$  has cardinality  $C(m + d - 1, d - 1) = \binom{m + d - 1}{d - 1}$ .*

**Lemma 2.4.** *We have*

$$\int_1^n (n - x)^{d-1} x \log x \, dx \geq \frac{1}{d(d+1)} n^{d+1} \log n - C_d n^{d+1}.$$

*Proof.* We claim for any  $m, \ell \geq 1$  that

$$\int_1^n (n - x)^m x^\ell \log x \, dx \geq \frac{m}{\ell + 1} \left[ \int_1^n (n - x)^{m-1} x^{\ell+1} \log x \, dx - \frac{n^{m+\ell+1}}{\ell + 1} \right].$$

We first note from integration by parts that

$$\int x^\ell \log x \, dx = \frac{x^{\ell+1}}{\ell + 1} \log x - \int \frac{x^\ell}{\ell + 1} dx = \frac{x^{\ell+1}}{\ell + 1} \log x - \frac{x^{\ell+1}}{(\ell + 1)^2} + C.$$

Now, using integration by parts again we obtain:

$$\begin{aligned} \int_1^n (n - x)^m x^\ell \log x \, dx &= (n - x)^m \left( \frac{x^{\ell+1}}{\ell + 1} \log x - \frac{x^{\ell+1}}{(\ell + 1)^2} \right) \Big|_1^n \\ &\quad + \int_1^n m(n - x)^{m-1} \left( \frac{x^{\ell+1}}{\ell + 1} \log x - \frac{x^{\ell+1}}{(\ell + 1)^2} \right) dx. \end{aligned}$$

We note that  $(n - x)^{m-1} x^{\ell+1} \leq n^{m+\ell}$  for  $x \in [1, n]$ . Thus, simplifying we get

$$\begin{aligned} \int_1^n (n - x)^m x^\ell \log x \, dx &\geq \frac{(n - 1)^m}{(\ell + 1)^2} + \frac{m}{\ell + 1} \int_1^n \left[ (n - x)^{m-1} x^{\ell+1} \log x - \frac{n^{m+\ell}}{\ell + 1} \right] dx \\ &\geq \frac{m}{\ell + 1} \left[ \int_1^n (n - x)^{m-1} x^{\ell+1} \log x \, dx - \frac{n^{m+\ell+1}}{\ell + 1} \right]. \end{aligned}$$

To prove the lemma we iterate the claim:

$$\begin{aligned} \int_1^n (n - x)^{d-1} x \log x \, dx &\geq \frac{d-1}{2} \left[ \int_1^n (n - x)^{d-2} x^2 \log x \, dx - \frac{n^{d+1}}{2} \right] \\ &\geq \frac{d-1}{2} \left[ \frac{d-2}{3} \left( \int_1^n (n - x)^{d-3} x^3 \log x \, dx - \frac{n^{d+1}}{3} \right) - \frac{n^{d+1}}{2} \right] \\ &\geq \dots \\ &\geq \frac{(d-1)!}{d!} \int_1^n x^d \log x \, dx - C'_d n^{d+1} \\ &= \frac{1}{d(d+1)} n^{d+1} \log n - C_d n^{d+1}. \quad \square \end{aligned}$$

We state without proof the following

**Lemma 2.5.** *Let  $m < n$  be integers and  $f : [m, n] \rightarrow [0, \infty)$  be a continuous function with exactly one local maximum in  $[m, n]$  and  $f(m) = f(n) = 0$ . Then, we have*

$$(4) \quad \left| \sum_{k=m}^n f(k) - \int_m^n f(x) dx \right| \leq \max_{m \leq x \leq n} f(x).$$

*Proof of Proposition 2.2.* We have

$$|\beta(\ell, \mathbf{m})| \geq \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \prod_{j_0=0}^{n-|\mathbf{j}|} |(\ell - j_0) + (\mathbf{m} - \mathbf{j}) \cdot \mathbf{x}|.$$

Since  $\mathbf{x}$  is Diophantine of order  $\mu$  we may find some  $\epsilon > 0$  such that  $\langle \mathbf{q} \cdot \mathbf{x} \rangle \geq \epsilon \|\mathbf{q}\|^{-\mu}$ . Using Lemma 2.1 we get

$$\begin{aligned} |\beta(\ell, \mathbf{m})| &\geq \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \prod_{j=-\ell}^{n-|\mathbf{j}|-\ell} |j - (\mathbf{m} - \mathbf{j}) \cdot \mathbf{x}| \\ &\geq \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \left( \frac{n-|\mathbf{j}|}{2e} \right)^{n-|\mathbf{j}|} \langle (\mathbf{m} - \mathbf{j}) \cdot \mathbf{x} \rangle \\ &\geq \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \left( \frac{n-|\mathbf{j}|}{2e} \right)^{n-|\mathbf{j}|} \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \epsilon \|\mathbf{m} - \mathbf{j}\|^{-\mu} \\ &= \left( \prod_{k=1}^n \prod_{|\mathbf{j}|=n-k, \mathbf{j} \neq \mathbf{m}} \left( \frac{k}{2e} \right)^k \right) \left( \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \epsilon \|\mathbf{m} - \mathbf{j}\|^{-\mu} \right). \end{aligned}$$

We set

$$A := \prod_{k=1}^n \prod_{|\mathbf{j}|=n-k, \mathbf{j} \neq \mathbf{m}} k^k, B := \prod_{k=1}^n \prod_{|\mathbf{j}|=n-k, \mathbf{j} \neq \mathbf{m}} (2e)^{-k}, C := \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \epsilon \|\mathbf{m} - \mathbf{j}\|^{-\mu}.$$

We now estimate each of  $A, B, C$  separately. Since the set  $\{\mathbf{j} \in \mathbb{Z}^d : |\mathbf{j}| \leq n\}$  has cardinality at most  $(n+1)^d$  and  $\|\mathbf{m} - \mathbf{j}\| \leq n$  for any  $|\mathbf{j}| \leq n$  we get that

$$C = \prod_{|\mathbf{j}| \leq n, \mathbf{j} \neq \mathbf{m}} \epsilon \|\mathbf{m} - \mathbf{j}\|^{-\mu} \geq \prod_{|\mathbf{j}| \leq n} \epsilon n^{-\mu} \geq \epsilon^{(n+1)^d} n^{-\mu(n+1)^d} \geq \epsilon^{(2n)^d} n^{-\mu(2n)^d}.$$

Thus,

$$(5) \quad \log C \geq -\mu 2^d n^d \log n + 2^d n^d \log \epsilon.$$

Using Lemma 2.3 together with the trivial bound we get

$$\begin{aligned} \log A &\geq \left( \sum_{k=1}^n \sum_{|\mathbf{j}|=n-k} k \log k \right) - n \log n \\ &= \left( \sum_{k=1}^n \binom{n-k+d-1}{d-1} k \log k \right) - n \log n \\ &\geq \left( \frac{1}{(d-1)!} \sum_{k=1}^n (n-k)^{d-1} k \log k \right) - n \log n. \end{aligned}$$

It is easy to see that the function  $f : [1, n] \rightarrow [0, \infty)$  given by  $f(x) = (n-x)^{d-1} x \log x$  satisfies Lemma 2.5. Thus, Lemma 2.4 and Lemma 2.5 give

$$\begin{aligned} \log A &\geq \frac{1}{(d-1)!} \left( \int_1^n (n-x)^{d-1} x \log x \, dx - \max_{1 \leq x \leq n} f(x) \right) - n \log n \\ &\geq \frac{1}{(d-1)!} \left( \frac{1}{d(d+1)} n^{d+1} \log n - C_d n^{d+1} - n^d \log n \right) - n \log n. \end{aligned}$$

Hence,

$$(6) \quad \log A \geq \frac{1}{(d+1)!} n^{d+1} \log n - 3C_d n^{d+1}.$$

On the other hand, since  $C(n-k+d-1, d-1) \leq \frac{n^{d-1}}{(d-1)!} + O(n^{d-2})$  for any  $k \in [1, n]$  we get

$$\begin{aligned} (7) \quad \log B &\geq - \sum_{k=1}^n \sum_{|\mathbf{j}|=n-k} k \log(2e) = - \sum_{k=1}^n \binom{n-k+d-1}{d-1} k \log(2e) \\ &\geq - \frac{2}{(d-1)!} n^{d+1} - O(n^d), \end{aligned}$$

where the implied constant depends  $d$  only. Thus, combining (5), (6) and (7) we arrive at

$$\log |\beta(\ell, \mathbf{m})| > \frac{1}{(d+1)!} n^{d+1} \log n - C_{d,\mu,\epsilon} n^{d+1}. \quad \square$$

*Proof of Theorem 1.2.* Let  $N = \dim \mathcal{P}_n - 1$ , so that  $N = \binom{n+d+1}{n} - 1$ .

Fix some  $P \in \mathcal{P}_n$  with  $\|P\|_K \leq 1$ . Define

$$P(\mathbf{z}) = \sum_{j_0+j_1+\dots+j_d \leq n} c(j_0, \mathbf{j}) z_0^{j_0} \dots z_d^{j_d} \text{ and } f(z) = P(e^z, e^{x_1 z}, \dots, e^{x_d z}),$$

where  $j_0, \dots, j_d \geq 0$ . Then,

$$f(z) = \sum_{j_0+j_1+\dots+j_d \leq n} c(j_0, \mathbf{j}) e^{(j_0+\mathbf{j} \cdot \mathbf{x})z}.$$

For any polynomial  $R(\lambda) = \sum_{j=0}^m c_j \lambda^j$  we introduce the differential operator

$$D_R = R\left(\frac{d}{dz}\right) = \sum_{j=0}^m c_j \frac{d^j}{dz^j}.$$

We note that for any  $a \in \mathbb{C}$  we have

$$(8) \quad D_R(e^{az})|_{z=0} = \sum_{j=0}^m c_j a^j = R(a).$$

To estimate  $c(\ell, \mathbf{m})$  we set

$$R_{\ell, \mathbf{m}}(\lambda) = \prod_{j_0+j_1+\dots+j_d \leq n, (j_0, \mathbf{j}) \neq (\ell, \mathbf{m})} (\lambda - (j_0 + \mathbf{j} \cdot \mathbf{x})) = \sum_{t=0}^N a_t \lambda^t.$$

For any  $\lambda \geq 0$  we have

$$\sum_{t=0}^N |a_t| \lambda^t \leq \prod_{j_0+j_1+\dots+j_d \leq n, (j_0, \mathbf{j}) \neq (\ell, \mathbf{m})} (\lambda + |j_0 + \mathbf{j} \cdot \mathbf{x}|) \leq (\lambda + n)^N.$$

From (8) we note that

$$D_{R_{\ell, \mathbf{m}}}(e^{(j_0 + \mathbf{j} \cdot \mathbf{x})z})|_{z=0} = \begin{cases} R_{\ell, \mathbf{m}}(\ell + \mathbf{m} \cdot \mathbf{x}) & \text{if } (j_0, \mathbf{j}) = (\ell, \mathbf{m}), \\ 0 & \text{if } (j_0, \mathbf{j}) \neq (\ell, \mathbf{m}). \end{cases}$$

Thus,

$$D_{R_{\ell, \mathbf{m}}}(f(z))|_{z=0} = c(\ell, \mathbf{m}) \beta(\ell, \mathbf{m})$$

where  $\beta(\ell, \mathbf{m})$  from (3).

On the other hand, using  $\|P\|_K \leq 1$  and Cauchy's inequality we get

$$(9) \quad |f^{(t)}(0)| \leq t! \leq N^t \text{ whenever } t \leq N.$$

This implies that

$$|D_{R_{\ell, \mathbf{m}}}(f(z))|_{z=0}| = \left| \sum_{t=0}^N a_t f^{(t)}(0) \right| \leq \sum_{t=0}^N |a_t| N^t \leq (N+n)^N.$$

Therefore,

$$\log(|c(\ell, \mathbf{m}) \beta(\ell, \mathbf{m})|) \leq N \log(N+n).$$

Using Proposition 2.2 we obtain

$$\begin{aligned} \log(|c(\ell, \mathbf{m})|) &\leq N \log(N+n) - \log |\beta(\ell, \mathbf{m})| \\ &\leq N \log(N+n) - \frac{1}{(d+1)!} n^{d+1} \log n + C_{\mathbf{x}, d} n^{d+1}. \end{aligned}$$

Since  $\|P\|_{\Delta^d} \leq \sum |c(j_0, \mathbf{j})| \leq (N+1) \max |c(j_0, \mathbf{j})|$  we deduce that

$$e_n(\mathbf{x}) \leq N \log(N+n) - \frac{1}{(d+1)!} n^{d+1} \log n + C_{\mathbf{x}, d} n^{d+1} + \log(N+1).$$

Finally, using

$$(10) \quad N = C(n + d + 1, d + 1) - 1 = \frac{n^{d+1}}{(d + 1)!} + O(n^d)$$

we obtain  $N \log(N + n) \leq N \log N + N = \frac{1}{d!} n^{d+1} \log n + O(n^{d+1})$ . Hence,

$$e_n(\mathbf{x}) \leq \frac{n^{d+1}}{(d - 1)!(d + 1)} \log n + O(n^{d+1}). \quad \square$$

### 3. LOWER ESTIMATE AND HAUSDORFF DIMENSION

We first start proving Theorem 1.1. It is essentially contained in the proof of [CP03, Proposition 1.3] as pointed out by D. Coman and for completeness we recall it here.

*Proof of Theorem 1.1.* Fix  $P \in \mathcal{P}_n(d+1)$  with  $\text{ord}(P(e^z, e^{x_1 z}, \dots, e^{x_d z}), 0) \geq N$ . We have  $P \not\equiv 0$  implies  $P(e^z, e^{x_1 z}, \dots, e^{x_d z}) \not\equiv 0$ . We let  $f(z) = \frac{1}{\|P\|_K} P(e^z, e^{x_1 z}, \dots, e^{x_d z})$  so that  $\|f\|_{\Delta^{d+1}} = 1$  then  $\max_{|z|=r} |f(z)| \geq r^N, r \geq 1$ . From (1) we get for any  $|z| = r$

$$r^N \leq E_n(\mathbf{x}) \exp(n \log^+ \max\{|e^z|, |e^{x_1 z}|, \dots, |e^{x_d z}|\}) \leq E_n(\mathbf{x}) e^{n C_0 r},$$

where  $C_0 = \max\{1, \|\mathbf{x}\|\}$ . Taking  $r = N/n$  we see that

$$N \log \frac{N}{n} \leq e_n(\mathbf{x}) + C_0 N.$$

Using (10) we have

$$N \log \frac{N}{n} = \frac{n^{d+1}}{(d - 1)!(d + 1)} \log n + O(n^d \log n),$$

which gives

$$e_n(\mathbf{x}) \geq \frac{n^{d+1}}{(d - 1)!(d + 1)} \log n - O(n^{d+1}). \quad \square$$

Now we prove Theorem 1.3 which provides us with the exceptional set of points  $\mathbf{x}$  that does not satisfy Theorem 1.2.

*Proof of Theorem 1.3.* Let  $\mathbf{x} \in W(d)$  and  $(\mathbf{q}_\ell)_{\geq 1}$  be a sequence satisfying

$$(11) \quad C(\ell) = \frac{-\log \langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle}{\|\mathbf{q}_\ell\|^{d+1} \log \|\mathbf{q}_\ell\|} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$

For a given  $\ell \geq 0$  we let  $n = d\|\mathbf{q}_\ell\|$  and  $p \in \mathbb{Z}$  be such that  $\langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle = |\mathbf{q}_\ell \cdot \mathbf{x} - p|$ . Since  $\|\mathbf{x}\| \leq 1$  we see that  $|p| \leq d\|\mathbf{q}_\ell\|$ . Then, the polynomial  $P$  given by

$$P(z_0, z_1, \dots, z_d) = z_0^p - \prod_{\ell=1}^d z_\ell^{q_\ell}$$



is in  $\mathcal{P}_n(d+1)$ . Clearly,  $\|P\|_{\Delta^{d+1}} = 2$ . Using  $|1 - e^\xi| \leq 2|\xi|$  for  $|\xi| \leq 1$  we get

$$|P(e^z, e^{x_1 z}, \dots, e^{x_d z})| = |e^{pz}(1 - e^{(\mathbf{q}_\ell \cdot \mathbf{x} - p)z})| \leq 2e^n \langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle,$$

whenever  $|z| \leq 1$ . Therefore,

$$E_n(\mathbf{x}) \geq \|P\|_{\Delta^{d+1}} / \|P\|_K \geq e^{-n} \frac{1}{\langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle}.$$

So, using (11) we get

$$\begin{aligned} e_n(\mathbf{x}) = \log E_n(\mathbf{x}) &\geq C(\ell) \|\mathbf{q}_\ell\|^{d+1} \log \|\mathbf{q}_\ell\| - n \\ &= C(\ell) \left(\frac{n}{d}\right)^{d+1} \log \frac{n}{d} - n. \end{aligned}$$

Thus,

$$\frac{e_n(\mathbf{x})}{n^{d+1} \log n} \geq \frac{1}{d^{d+1}} C(\ell) - \frac{1}{n} \rightarrow \infty \text{ as } \ell \rightarrow \infty. \quad \square$$

It remains to give the proof of Theorem 1.4.

*Proof of Theorem 1.4.* We will use ubiquitous systems introduced in [DRV90] as a method of computing Hausdorff dimension of lim-sup sets. We consider the family  $\mathcal{R} = \{R(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}_{\geq 0}^d\}$  where for any  $\mathbf{q} \in \mathbb{Z}^d$  we set  $R(\mathbf{q}) := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{q} \cdot \mathbf{x} \in \mathbb{Z}\}$ . Let  $\psi : \mathbb{N} \rightarrow [0, 1]$  be a decreasing function converging to 0 at the infinity. Define

$$\Lambda(\mathcal{R}; \psi) = \left\{ \mathbf{x} \in [-1, 1]^d : \text{dist}(\mathbf{x}, R(\mathbf{q})) < \psi(\|\mathbf{q}\|) \text{ for infinitely many } R(\mathbf{q}) \right\},$$

where  $\text{dist}(\mathbf{x}, S) = \inf_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|$ . For any such  $\psi$ , we will prove that the Hausdorff dimension of  $\Lambda(\mathcal{R}; \psi)$  is at least  $d - 1$ . Then, for  $\psi(n) = n^{-n^{d+2}}$  we will show that  $\Lambda(\mathcal{R}; \psi) \subset W(d)$  which will finish the proof.

Let  $I^d$  denote the hypercube  $[-\frac{1}{2}, \frac{1}{2}]^d$  of unit length. It is well-known (see e.g. [Do93]) that the family  $\{R(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}^d\}$  is *ubiquitous with respect to*  $\rho(Q) := dQ^{-1-d} \log Q$  in the sense that

$$\left| I^d \setminus \bigcup_{1 \leq \|\mathbf{q}\| \leq N} B(R(\mathbf{q}); \delta(N)) \right| \rightarrow 0 \text{ as } N \rightarrow \infty,$$

$B(R(\mathbf{q}); \delta) = \{\mathbf{x} \in \mathbb{R}^d : \text{dist}(\mathbf{x}, R(\mathbf{q})) < \delta\}$ . However, it is not clear if the family  $\mathcal{R} = \{R(\mathbf{q}) : \mathbf{q} \in \mathbb{Z}_{\geq 0}^d\}$  is ubiquitous with respect to the same  $\rho$ . However, for our purposes we do not need to try to optimize  $\rho$ . Simply consider the constant function  $\rho \equiv 1$ , then for  $\mathbf{q} = (0, \dots, 0, 1)$  we have  $I^d \subset B(R(\mathbf{q}); 1)$  so that  $\mathcal{R}$  is ubiquitous w.r.t 1. Since  $\gamma := \limsup_{Q \rightarrow \infty} \frac{\log \rho(Q)}{\log \psi(Q)} = 0$ , it follows from [DRV90, Theorem 1] that the Hausdorff dimension of  $\Lambda(\mathcal{R}; \psi)$  is at least  $\dim \mathcal{R} + \gamma \text{codim } \mathcal{R} = d - 1$ .

We now claim that  $\Lambda(\mathcal{R}; \psi) \subset W(d)$  when  $\psi(n) = n^{-n^{d+2}}$ . For  $\mathbf{x} \in \Lambda(\mathcal{R}; \psi)$  let  $(\mathbf{q}_\ell)_{\ell \geq 1}$  denote the sequence such that  $\text{dist}(\mathbf{x}, R(\mathbf{q}_\ell)) < \psi(\|\mathbf{q}_\ell\|)$  and all

$R(\mathbf{q}_\ell)$  are distinct. Then, for any  $\mathbf{q} \in \mathbb{Z}^d$  we have  $\mathbf{q} \cdot \mathbf{x} \notin \mathbb{Z}$  which means  $\{1, x_1, x_2, \dots, x_d\}$  is linearly independent over  $\mathbb{Z}$ . Let  $\mathbf{y} \in R(\mathbf{q}_\ell)$  be such that  $\|\mathbf{x} - \mathbf{y}\| < \psi(\|\mathbf{q}_\ell\|)$ . We choose  $p \in \mathbb{Z}$  with  $\mathbf{q}_\ell \cdot \mathbf{y} = p$ . Then,

$$\langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle \leq \|\mathbf{q}_\ell \cdot (\mathbf{x} - \mathbf{y}) + \mathbf{q}_\ell \cdot \mathbf{y} - p\| \leq \|\mathbf{q}_\ell\| \|\mathbf{x} - \mathbf{y}\| < \psi(\|\mathbf{q}_\ell\|).$$

Hence,  $\mathbf{x} \in W(d)$  as  $\frac{-\log(\langle \mathbf{q}_\ell \cdot \mathbf{x} \rangle)}{\|\mathbf{q}_\ell\|^{d+1} \log \|\mathbf{q}_\ell\|} \geq \|\mathbf{q}_\ell\|$  and  $\|\mathbf{q}_\ell\| \rightarrow \infty$  with  $\ell \rightarrow \infty$ .  $\square$

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